

# The Minimum Perfect Matching in Pseudo-dimension $0 < q < 1$

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## ABSTRACT

It is known that for  $K_{n,n}$  equipped with i.i.d.  $\exp(1)$  edge costs, the minimum total cost of a perfect matching converges to  $\pi^2/6$  in probability. Similar convergence has been established for all edge cost distributions of *pseudo-dimension*  $q \geq 1$ , such as Weibull( $1, q$ ) costs. In this paper we extend those results to all  $q > 0$ , confirming the Mézard-Parisi conjecture in the last remaining applicable case.

**Keywords**— matching, mean field, replica symmetry, random graph, pseudo-dimension

## 1 INTRODUCTION

There has been substantial interest over the past few decades in the minimum matching problem: Given a graph  $G$ , and a positive cost associated to each edge of  $G$ , we want to find a perfect matching of minimal total cost  $M(G)$ . Of special interest is minimum matching on the complete bipartite graph  $K_{n,n}$  on  $n + n$  vertices with random edge costs given by independent  $\exp(1)$ -variables, sometimes referred to as the *random assignment* problem.

For this graph model, the lower bound 1 on the cost  $M(K_{n,n})$  of minimum matching is trivial, and the upper bound 3 was established by Walkup [9] by finding perfect matchings using only fairly cheap edges. This was later improved to 2 by Karp [3]. Mézard & Parisi [6] conjectured that  $M(K_{n,n})$  converges in probability to  $\pi^2/6$ , based on heuristic replica symmetry calculations. Aldous [1] proved that the limit exists, and later confirmed the conjecture [2]. Both of these papers used what is sometimes called the objective method, and worked with matchings on an infinite limit object. Parisi [5] further conjectured the more precise result that  $\mathbb{E}[M(K_{n,n})] = \sum_{k=1}^n k^{-2}$ . This was later established independently by Nair, Prabhakar and Shaw [8] and Wästlund [12], both using inductive proofs. This was later simplified by Wästlund [10]. Salez and Shah [7] gave yet another proof, using the objective method to analyze the behavior of belief propagation on the limit object.

A more comprehensive overview of the existing literature and related problems can be found in a survey paper by Krokmal and Pardalos [4].

A natural question is whether these results extend to other edge cost distributions. A random graph with i.i.d. edge costs given by the cumulative distribution function  $\ell$  is said to be of *pseudo-dimension*  $q$  if  $\ell(x) \sim x^q$  for small  $x$ . The exponential distribution is of pseudo-dimension 1,

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and more generally a Weibull distribution with shape parameter  $q$  (i.e. the  $(1/q)$ :th power of an exponential variable) is of pseudo-dimension  $q$ . Mézard and Parisi [6] considered these distributions for real positive  $q$ , but most focus since then has been on the special case  $q = 1$ .

The motivation for the term pseudo-dimension is this: For  $q \in \mathbb{N}$ , a geometric graph model is given by embedding the vertices as  $n$  random points in a hypercube  $[0, 1]^q$ , and setting the edge costs to be the corresponding Euclidean distances. The mean field approximation (i.e. the graph model given by independently rerandomizing each edge cost) of a geometric graph model of dimension  $q$  is a graph model of pseudo-dimension  $q$ .

Let  $K_{n,n}^q$  denote  $K_{n,n}$  equipped with independent Weibull(1,  $q$ ) edge costs. The cost of a minimum matching on  $K_{n,n}^q$  can be shown to be of order  $n^{1-1/q}$ , by a minor modification of [9]. This suggests studying the quantity  $n^{-1+1/q}M(K_{n,n}^q)$ . Does it converge in probability to a constant for any  $q > 0$ ? This question was answered in the affirmative for  $q \geq 1$  in 2011 by Wästlund [11], but it remained open for  $0 < q < 1$ . Our main result is the following theorem.

**THEOREM 1:** *For every  $q > 0$ , there exists a real number  $\beta(q)$  such that  $n^{-1+1/q}M(K_{n,n}^q) \rightarrow \beta(q)$  in probability as  $n \rightarrow \infty$ .*

This theorem should also hold in somewhat greater generality, i.e. other graphs than  $K_{n,n}$ . We speculate that similar results hold for every graph  $G$  which can be thinned to a graph  $G'$  of unbounded average degree such that  $G'$  is indistinguishable from an Erdős-Renyi graph. Or, put in a more general way:

**CONJECTURE 2:** *Let  $G_n$  be a sequence of graphs which admits a graphon limit  $\mathcal{G}$ , and whose edges are given weights following some distribution of pseudo-dimension  $q$ . If there exists  $\varepsilon > 0$  such that the measure on  $[0, 1]^2$  given by  $\mathcal{G}$  has density at least  $\varepsilon$  everywhere, then  $n^{-1+1/q}M(G_n)$  converges in probability to a constant (depending only on  $q$  and  $\mathcal{G}$ ).*

The conjecture doesn't quite cover all the cases where one would expect the result to hold. Another example would be complete  $m$ -partite graphs for any  $m > 2$ .

## 2 NOTATION AND DEFINITIONS

In what follows, we will use  $\int$  as short-hand for  $\int_{-\lambda/2}^{\lambda/2}$  if no other integration limits are given. We will assume  $q, \lambda > 0$  are fixed, and often suppress dependence on them in our notation. Since Theorem 1 is already known to be true for  $q \geq 1$ , we will for convenience restrict our attention to  $q < 1$ . Unless otherwise stated, all functions considered will be real-valued functions on  $[-\lambda/2, \lambda/2]$ . For  $f$  and  $g$  functions, we will henceforth use  $f \leq g$  to mean that  $f(z) \leq g(z)$  for all  $z$  in their domain. For  $x \in \mathbb{R}$ , we let  $x_+ := \max(x, 0)$ .

Let  $T_\lambda^q$  be an edge-weighted Galton-Watson tree, with offspring given by an inhomogeneous Poisson process on  $[0, \lambda]$  with intensity  $qt^{q-1}$  at time  $t$ . Every event in this process gives rise to an offspring, and the corresponding edge weight is the arrival time. (Note that  $T_\lambda^q$  is the local limit of the graph given by removing all edges of cost more than  $\lambda$  from  $K_{n,n}^q$ .) Furthermore, we will let  $|G|$  denote the number of *edges* of a graph  $G$  and we will consider the edges of  $T_\lambda^q$  to be directed away from the root  $\phi$ . By *path* we will mean a directed path away from the root. If  $u$  is the parent of  $v$ , we write  $u \rightarrow v$ . For any edge  $uv$  we let  $\ell(u, v)$  denote its cost.

### 3 PROOF STRATEGY

Following the work by Wästlund [11], we will not deal with matchings directly, but instead study the game *Exploration*. This zero-sum, perfect information game is played in the following way: On an edge-weighted tree  $T$ , Alice and Bob takes turns picking the next edge of a self-avoiding walk starting from the root  $\phi$ . When it is a player's turn (Alice's, say), and the current vertex is  $u$ , she can take one of two actions:

- (i) Pick any neighbour  $v$  of  $u$  that has not already been visited, and pay Bob the cost of the edge  $(u, v)$ . Bob then continues the game from  $v$ .
- (ii) Quit the game, and pay Bob a penalty of  $\lambda/2$ , for some fixed positive parameter  $\lambda$ .

The payoff for Alice, once the game has finished, is the total amount Bob has payed to her minus the total amount she has payed to Bob. Each player's aim is to maximize their payoff. Note that it is never optimal for Alice to choose an edge more expensive than  $\lambda$ , since even in the best case scenario where Bob quits immediately afterwards, Alice will have lost more than the penalty of  $\lambda/2$  it would have cost her to quit instead. Similarly, Bob will never want to pick such an edge either. Furthermore, the game is entirely non-random on any fixed tree.

By the game value of  $u \in V(T)$  we will mean the value of moving *to* the vertex  $u$ . A function  $f : V(T) \rightarrow [-\lambda/2, \lambda/2]$  is called a game valuation if for every vertex  $u$  it satisfies

$$f(u) = \begin{cases} \min_{v: u \rightarrow v} [\ell(u, v) - f(v)], & \text{if } u \text{ has at least one offspring} \\ \lambda/2, & \text{if } u \text{ is a leaf} \end{cases}$$

There exists a unique valuation if the tree is finite; it is recursively defined and the recursion ends at the leaves. For infinite graphs, however, there may be more than one valuation.

Wästlund [11] established that for any  $q > 0$  the limit of  $n^{-1+1/q} M(K_{n,n}^q)$  exists *if* the game valuation of Exploration played on  $T_\lambda^q$  is well defined (unique) for any  $\lambda$ . He proceeded to prove it was indeed well defined for  $q \geq 1$ , but the proof did not work for  $0 < q < 1$ . Therefore, in order to prove Theorem 1 it suffices to show that the following proposition (Proposition 2.8 in that paper) holds in our current setting too.

**PROPOSITION 3:** *There is almost surely only one valuation.*

The set of all possible game valuations consistent with  $T_\lambda^q$  can be given a bounded lattice ordering in a natural way. The unique maximum and minimum in this lattice correspond to the most optimistic valuations of the game Alice and Bob could possibly have ( $f_A$  and  $f_B$ , respectively). They are such that  $f_A(u) \leq f_B(u)$  for  $u$  at even distance from the root, and with the equality reversed for  $u$  at odd distance. To prove Proposition 3 it suffices to show that these two are identical [11, p.1072].

The main idea of Wästlund's proof was to view  $f_A$  as known and  $f_B$  as unknown, and use  $f_A$  to predict which moves Bob could not possibly consider to be optimal. It turns out that the difference  $|f_A - f_B|$  is non-decreasing along the game path if each player plays optimally with respect to their valuation, and for any  $\delta > 0$  the moves by Bob will be within  $\delta$  of being optimal w.r.t.  $f_A$  eventually – after the vertex  $u$ , say, has been reached. Such moves were called  *$\delta$ -reasonable*. For  $q \geq 1$ ,  $\delta$  can be chosen sufficiently small so that the tree rooted in  $u$  consisting of all  $\delta$ -reasonable moves is a Galton-Watson tree with branching number bounded away from 1. This tree is a.s. finite and is guaranteed to contain the game path from  $u$  onwards, so the game will a.s. end after a finite number of moves, whence  $f_A$  and  $f_B$  must agree.

We will employ a similar strategy to extend the result to all  $q > 0$ , but we will need three additional tricks. First, by keeping more careful track of the game history from the root  $\phi$  to a vertex  $u$ , we will be able to rule out more move options from  $u$ . Instead of considering a single move as reasonable, we consider entire game paths, and call a game path  $(u, t)$ -reasonable if the sum of all of Bob's deviations from  $f_A$ , starting from  $u$ , is at most  $t$ . We show that the actual game path is  $(\phi, 2\lambda)$ -reasonable, and bound the size of the tree  $\Delta_t(u)$  consisting of all  $(u, t)$ -reasonable paths.

Second, instead of showing that  $\Delta_t(u)$  has branching number uniformly bounded away from 1 for some  $t$  and  $u$ , we proceed differently. We recursively bound the size of sub-trees of  $\Delta_t(u)$ , using that the sub-tree of  $T_\lambda^q$  rooted in any vertex has the same distribution as the whole  $T_\lambda^q$ .

The third trick is the crucial step: construct a positive linear operator that describes this recursion, and bound its operator norm away from 1 in order to show that the recursion ends. To accomplish this we will need good control of the distribution of  $f_A(\phi)$ , and a fair bit of calculus.

## 4 PROPOSITIONS AND LEMMAS

As mentioned earlier it suffices to prove that Proposition 3 holds for  $0 < q < 1$ . We will use Proposition 6 and Lemmas 4 and 5 to prove Proposition 3. Proposition 6, in turn, uses Lemmas 7 to 9. In this section we simply state the lemmas and propositions, proofs will then appear in the next section.

For  $v \in T_\lambda^q - \{\phi\}$ , let  $\delta(v)$  be how far from  $f_A$ -optimal it is to move to  $v$  from its parent. In other words, if  $u$  is the parent of  $v$ , then  $\delta(v) := \ell(u, v) - f_A(u) - f_A(v)$ . (Clearly,  $\delta \geq 0$ .)

For  $u$  at even distance from the root we say that a finite path  $u \rightarrow u_1 \rightarrow \dots \rightarrow u_n$  is  $(u, t)$ -reasonable if  $\sum_{i=1}^n \delta(u_i) \leq t$  and  $\delta(u_i) = 0$  for all odd  $i$ . For infinite paths the definition is analogous.

LEMMA 4: *The game path is  $(\phi, 2\lambda)$ -reasonable.*

We will modify slightly how we generate the tree  $T = T_\lambda^q$ , so that we condition on  $f_A$  in each step, and label each vertex  $u$  with  $f_A(u)$ . Consider the distribution of  $f_A(\phi)$ , let  $F_A(z) := \mathbb{P}(f_A(\phi) \geq z)$  be its anti-cdf, and let  $F_B$  be defined analogously. In a slight abuse of notation, we will sometimes refer to  $F_A$  and  $F_B$  as the corresponding probability measures on  $[-\lambda/2, \lambda/2]$ .

Let the  $\ell f$ -square be the set  $[0, \lambda] \times [-\lambda/2, \lambda/2]$ , where we consider a vertex  $v$  with parent  $u$  to correspond to the point  $(\ell(u, v), f_A(v))$ . Let  $m_\ell$  be the measure on  $[0, \lambda]$  given by  $qt^{q-1}$  times the Lebesgue measure. We equip the  $\ell f$ -square with two measures:  $\mu_A$  is the product measure of  $m_\ell$  and  $F_A$ , and  $\mu_B$  is the product measure of  $m_\ell$  and  $F_B$ .

We will construct a new edge-weighted tree  $\tilde{T}$  with a real-valued vertex labelling, where the offspring of a vertex is given by picking random points in the  $\ell f$ -square. We construct  $\tilde{T}$  inductively in the following way:

INITIALIZE Give  $\phi$  a random vertex labelling, following the distribution  $F_A$ , and mark  $\phi$  as active.

REPEAT Pick any active vertex  $u$  (if there is no active vertex, end) and let  $z := f_A(u)$ . If  $u$  is at odd distance from the root, generate points in the  $\ell f$ -square by a Poisson random measure given by the measure  $\chi_{\{\ell-f > z\}} \mu_A$ . Also generate an extra point by choosing a point on the diagonal  $\ell - f = z$  according to the normalized marginal measure.

If on the other hand  $u$  is at even distance from the root, do the same thing but with  $\mu_B$  instead of  $\mu_A$ .

These points will be the offspring of  $u$ : for every point  $(\ell, f)$  we get, add a child  $v$  to  $u$ , let  $\ell(u, v) := \ell$ , and label  $v$  with  $f$ , and mark  $v$  as active. Finally, mark  $u$  as inactive.

LEMMA 5: *If we label the vertices of  $T$  using the function  $f_A$ , then  $T \sim \tilde{T}$ .*

Since  $T \sim \tilde{T}$ , we will couple them so that  $T = \tilde{T}$ . This is why it makes sense to view Bob's moves as unknown: If we let  $N_v$  be the graph induced by some vertex  $v \in T$  and its offspring (and inheriting the vertex and edge labels of  $T$ ), then Alice's move from  $v$  is measurable w.r.t. the  $\sigma$ -algebra generated by  $N_v$ , but Bob's move may not be measurable. It may in fact not be measurable w.r.t. any  $\sigma$ -algebra generated by a finite subgraph.

Let  $\Delta_t(u)$  be the union of all  $(u, t)$ -reasonable paths. Our aim is to bound the size of  $\Delta_t(u)$ , and we will do this by finding a bound for the size of  $k$ -level truncations  $\Delta_t^k(u)$ . Conditioned on  $f_A(u)$ , the distribution of  $\Delta_t^k(u)$  is the same for every  $u$  at even distance from the root, so we let

$$R_t^k(z) := \mathbb{E}[|\Delta_t^k(\phi)| | f_A(\phi) = z]. \quad (1)$$

PROPOSITION 6: *There exists a family of continuous functions  $\psi_t$  s.t.  $R_t^{2k}(z) < \psi_t(z)$  for all  $z \in [-\lambda/2, \lambda/2]$ ,  $t \in [0, 2\lambda]$ , and  $k \in \mathbb{N}$ , and satisfying  $\sup_{z,t} \psi_t(z) < \infty$ . Hence  $\Delta_\lambda$  is almost surely finite.*

To prove Proposition 6 we will need the following lemmas. The technical Lemmas 7 and 8 mostly concerns deriving bounds, and their proofs are to a large extent calculus exercises. Lemma 9, on the other hand, is central to this paper.

LEMMA 7:  *$F_A(z) = \mathbb{P}(f_A(\phi) \geq z)$  and  $F_B(z) = \mathbb{P}(f_B(\phi) \geq z)$  exist and are continuously differentiable, with  $F'_A$  satisfying*

$$F'_A(z) = -F_A(z) \cdot \left( F_B(\lambda/2) q(z + \lambda/2)^{q-1} - \int q(z+t)_+^{q-1} F'_B(t) dt \right). \quad (2)$$

Furthermore,  $\int q(z+t)_+^{q-1} F'_A(t) dt$  is continuous in  $z$ , and for some constant  $\alpha$  and all  $-\lambda/2 < z < \lambda/2$ , we have the bounds

$$-F'_A(z) \leq \alpha(\lambda/2 - |z|)^{q-1}, \quad (3)$$

$$- \int_{\max(-\lambda/2, -z)}^{\min(\lambda/2, \lambda-z)} q(z+t)_+^{q-1} F'_A(t) dt \leq \alpha \max((z + \lambda/2)^{2q-1}, |z - \lambda/2|^{q-1}). \quad (4)$$

Equation (4) also holds for  $\lambda/2 < z \leq 3\lambda/2$ , and analogous results hold for  $F'_B$ .

We will use eq. (4) in another form: Define  $J_A^z$  (or  $J_B^z$ ) as the marginal measure w.r.t.  $\mu_A$  (or  $\mu_B$ ) of the line  $\ell - f = z$  in the  $\ell f$ -square. It then satisfies

$$J_A^z \leq \alpha \max((z + \lambda/2)^{2q-1}, |z - \lambda/2|^{q-1}). \quad (5)$$

LEMMA 8: *For  $|z| < \lambda/2$ , let  $\rho_A^z(t)$  be the density in the  $\ell f$ -square along the diagonal  $\ell - f = z$ :*

$$\rho_A^z(t) := q(z+t)_+^{q-1} \cdot (-F'_A(t)) \quad (6)$$

and let the positive linear operator  $L_A$  on  $\mathcal{C}([-\lambda/2, \lambda/2])$  and the function  $I_A$  be defined by

$$L_A(h)(z) := \frac{\int h(t) \rho_A^z(t) dt}{q(z + \lambda/2)^{q-1} F_A(\lambda/2) + \int \rho_A^z(t) dt} \quad (7)$$

$$I_A(z) := \frac{\int \rho_A^z(t) dt}{q(z + \lambda/2)^{q-1} F_A(\lambda/2) + \int \rho_A^z(t) dt} \quad (8)$$

on  $(-\lambda/2, \lambda/2)$ , and by their continuous extensions at  $\pm\lambda/2$ . Let also  $\rho_B^z, L_B$  and  $I_B$  be defined similarly. Let  $u, v$  be such that  $\phi \rightarrow u \rightarrow v$  are  $f_A$ -optimal moves. Then the following holds:

$$L_B \circ L_A(R_t^k)(z) = \mathbb{E}[\Delta_t^k(v) | f_A(\phi) = z]. \quad (9)$$

Furthermore,  $I_A$  satisfies these properties: (i)  $I_A$  is continuous, (ii)  $I_A(z) < 1$  for  $z \in [-\lambda/2, \lambda/2)$ , and (iii)  $I_A(\pm\lambda/2)$  are well defined by continuous extension. Analogous statements hold for  $I_B$ .

LEMMA 9:  $\|L_B \circ L_A\| < 1$ , where  $\|\cdot\|$  is the operator norm given by the  $\infty$ -norm on  $\mathcal{C}([-\lambda/2, \lambda/2])$ .

## 5 PROOFS

PROOF OF LEMMA 4. Let  $P$  be the game path. Pick any length 2 path  $(u \rightarrow v \rightarrow w) \subseteq P$ , such that  $u$  is at even distance from  $\phi$ .

Since  $u$  is at even distance from the root, it will be Alice's turn to move from  $u$ . She will choose the  $f_A$ -optimal move, i.e. she will move to a child  $v$  of  $u$  such that  $f_A(u) = \ell(u, v) - f_A(v)$ . In other words,  $\delta(v) = 0$ . This move may not be  $f_B$ -optimal, so  $f_B(u) \leq \ell(u, v) - f_B(v)$ . Thus<sup>1</sup>

$$f_A(u) - f_B(u) \geq [\ell(u, v) - f_A(v)] - [\ell(u, v) - f_B(v)] = f_B(v) - f_A(v).$$

Then it will be Bob's turn to move from  $v$ . He will choose the  $f_B$ -optimal move, i.e. he will move to a child  $w$  of  $v$  such that  $f_B(v) = \ell(v, w) - f_B(w)$ . Since this may not be the  $f_A$ -optimal move,  $f_A(v) = \ell(v, w) - f_A(w) - \delta(w)$ . Thus

$$f_B(v) - f_A(v) = [\ell(v, w) - f_B(w)] - [\ell(v, w) - f_A(w) - \delta(w)] = f_A(w) - f_B(w) + \delta(w),$$

and together with the move  $u \rightarrow v$  this gives that

$$f_A(u) - f_B(u) \geq f_A(w) - f_B(w) + \delta(w) = f_A(w) - f_B(w) + \delta(v) + \delta(w)$$

So if we take any even-length sub-path  $\phi = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_{2n}$  of the game path, and apply the same argument with  $(u, v, w) = (u_{2i-2}, u_{2i-1}, u_{2i})$ , for  $i \in [n]$ , we get that

$$\begin{aligned} f_A(\phi) - f_B(\phi) &\geq f_A(u_2) - f_B(u_2) + \delta(u_1) + \delta(u_2) \\ &\geq f_A(u_4) - f_B(u_4) + \delta(u_1) + \delta(u_2) + \delta(u_3) + \delta(u_4) \\ &\vdots \\ &\geq f_A(u_{2n}) - f_B(u_{2n}) + \sum_{i=1}^{2n} \delta(u_i) \end{aligned}$$

Rearranging gives

$$\sum_{i=1}^{2n} \delta(u_i) \leq f_A(\phi) - f_B(\phi) - f_A(u_{2n}) + f_B(u_{2n}) \leq 2\lambda,$$

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<sup>1</sup>A similar argument is used in [11, p.1076] to show that the difference  $f_A(u_{2k}) - f_B(u_{2k})$  is monotone.

where the last inequality holds because  $f_A(\bullet), f_B(\bullet) \in [-\lambda/2, \lambda/2]$ . Since  $\delta(u_{2n+1}) = 0$ , we also have that  $\sum_{i=1}^{2n+1} \delta(u_i) \leq 2\lambda$ . This means that for any finite subset  $P' \subseteq P$ , we have that  $\sum_{u \in P'} \delta(u) \leq 2\lambda$ . Thus  $\sum_{u \in P} \delta(u) \leq 2\lambda$ , and  $P$  must be a  $(\phi, 2\lambda)$ -reasonable path. ■

PROOF OF LEMMA 5. Recall that when we construct  $\tilde{T}$  we first generate  $f_A(\phi)$ , and then build the tree level by level conditioned on the game value of each new parent.

In the tree  $T$ , the offspring of the root, and the corresponding edge costs, are given by a Poisson random measure on  $[0, \lambda]$  given by the measure  $m_\ell$ . Each offspring  $u$  then has game value  $f_A(u) \sim f$ , for  $f$  drawn from the probability measure  $F_B$ . This is equivalent to drawing edge cost and game value simultaneously by a Poisson random measure on the  $\ell f$ -square given by the measure  $\mu_B$ .

In the tree  $\tilde{T}$ , on the other hand, we generate the game value of the root first, and then generate the rest of the tree conditioning on that value. What does it mean for the Poisson random measure to condition on  $f_A(\phi) = z$ ? The function  $f_A$  is recursively defined by  $f_A(\phi) = \min(\lambda/2, \min_{u: \phi \rightarrow u} (\ell(\phi, u) - f_A(u)))$ , so  $f_A(\phi) = z$  precisely when the Poisson random measure gives no points  $(\ell, f)$  with  $\ell - f < z$  and point  $(\ell, f)$  with  $\ell - f = z$ . The event  $\{f_A(\phi) = z\}$  is independent from the number of points with  $\ell - f > z$ .

Note that for a vertex  $u$  at even distance from the root,  $f_A(u)$  has the same distribution as  $f_A(\phi)$ , since the subtree rooted in  $u$  has the same distribution as the entire tree  $T$  and Alice is the player to move both from  $\phi$  and  $u$ . If, on the other hand,  $u$  is at odd distance from the root, Bob is the player to move from  $u$  and  $f_A(u)$  has the same distribution as  $f_B(\phi)$ .

Thus the distribution of the zeroth and first level of  $T$  and  $\tilde{T}$  are identical. Furthermore, the vertex labels of  $\tilde{T}$  also follow the same distribution as  $f_A$  on  $T$ . But the same argument applies for any level of the two trees, so  $T \sim \tilde{T}$ . ■

PROOF OF LEMMA 7. We will consider some bounded non-increasing function  $0 < G \leq 1$ , derive the desired results for  $G$ , and then show that  $F_A$  and  $F_B$  satisfy those same conditions. This proof hinges on the relationships  $F_A = V(F_B)$  and  $F_B = V(F_A)$  [11, p.1077], where  $V$  is the non-linear operator defined by

$$V(G)(z) = \exp \left( - \int q(z+t)_+^{q-1} G(t) dt \right).$$

Note that  $V(G)(-\lambda/2) = 1$ , while  $0 < V(G)(z) < 1$  for all other  $z$ . Bounded monotone functions have bounded variation, so we can integrate with respect to the measure  $dG$ , in the sense of a Riemann-Stieltjes integral. Let  $dG$  also have a point measure of mass  $-G(\lambda/2)$  at  $\lambda/2$ , as if  $G$  had a jump discontinuity there.

CLAIM: *Given this definition of  $dG$ ,*

$$\frac{d}{dz} V(G)(z) = V(G)(z) \cdot \int q(z+t)_+^{q-1} dG(t). \quad (10)$$

To verify eq. (10), start by integrating  $\int_{-z}^{\lambda/2} q(z+t)^{q-1} dG(t)$  from  $z = -\lambda/2$  to  $x$ :

$$\int_{-\lambda/2}^x \int_{-z}^{\lambda/2} q(z+t)_+^{q-1} dG(t) dz = \iint_{\substack{s-t \leq x \\ s, t \in [-\lambda/2, \lambda/2]}} q s_+^{q-1} dG(t) ds = \ln(V(G)(x))$$

So by the fundamental theorem of calculus,  $\ln(V(G)(z))$  is differentiable, with derivative given by  $\frac{d}{dz} \ln(V(G)(z)) = \int_{-z}^{\lambda/2} q(z+t)^{q-1} dG(t)$ . This implies that  $V(G)$  is also differentiable, with derivative given by eq. (10), proving the claim.

CLAIM: Let the function  $g$  be defined by

$$g(z) := (\lambda/2 - |z|)_+^{q-1} \quad (11)$$

Then there exists a constant  $a > 0$  such that if  $G$  is differentiable and satisfies  $-G' \leq a \cdot g$ , then  $-(V(G))' \leq a \cdot g$  also.

We need to calculate (and then estimate)  $\frac{d}{dz} V(G)(z)$ . Using eq. (10), we find that

$$\frac{d}{dz} V(G)(z) = -V(G)(z) \cdot \left( G(\lambda/2) \cdot q(\lambda/2 + z)^{q-1} - \int_{-z}^{\lambda/2} G'(t) \cdot q(z+t)^{q-1} dt \right). \quad (12)$$

Setting  $G = F_A$  or  $F_B$  gives eq. (2). Next, we will establish eqs. (3) and (4). The integrand on the right hand side of eq. (12) has a pole at  $t = -z$ , and the bound  $g$  has a pole at  $t = \lambda/2$ . For some positive parameter  $r < \min(\lambda/4, 2^{-4/q})$ , we will deal separately with the cases when the poles are within  $2r$  of each other and when they are further apart. We will establish that the following inequality holds in both cases:

$$-\frac{d}{dz} V(G)(z) < g(z) \cdot (2q + 4ar^q) + qr_+^{q-1}, \text{ for all } z \quad (13)$$

from which it follows that  $-\frac{d}{dz} V(G)(z) < a \cdot g(z)$  for all  $z$  by letting  $a > \max(8q, \lambda/r)$ .

CASE 1:  $-\lambda/2 \leq z \leq -\lambda/2 + 2r$

We apply the bound  $-G'(t) \leq a \cdot g(t)$ , and use that the resulting integrand is symmetric around  $t = -z/2 + \lambda/4$ :

$$\begin{aligned} -\int_{-z}^{\lambda/2} G'(t) \cdot q(z+t)^{q-1} dt &\leq a \int_{-z}^{\lambda/2} q(\lambda/2 - t)^{q-1} (z+t)^{q-1} dt \\ &\leq 2a(z/2 + \lambda/4)^{q-1} \cdot \int_0^{z/2 + \lambda/4} qs^{q-1} ds \\ &= 2^{2-2q} a(z + \lambda/2)^{2q-1} \end{aligned} \quad (14)$$

$$< 4ar^q g(z). \quad (15)$$

We will later be using the tighter bound in eq. (14), but for now eq. (15) suffices. Again using eq. (12), this gives a bound on  $\frac{d}{dz} V(G)(z)$ :

$$-\frac{d}{dz} V(G)(z) \leq G(z) \cdot \left( G(\lambda/2) \cdot q(\lambda/2 + z)_+^{q-1} + 4r^q g(z) \right) \leq g(z) \cdot (q + 4ar^q)$$

which is less than the bound from eq. (13).

CASE 2:  $-\lambda/2 + 2r \leq z \leq \lambda/2$

We use the bound  $-G'(t) \leq a \cdot g(t)$  for  $-z < t < -z + r$ .

$$-\int_{-z}^{\lambda/2} G'(t) \cdot q(z+t)^{q-1} dt \leq a \cdot \int_{-z}^{-z+r} g(t) \cdot q(z+t)^{q-1} dt - \int_{-z+r}^{\lambda/2} G'(t) \cdot q(z+t)^{q-1} dt \quad (16)$$



If  $g(t)$  is larger than  $g(-z)$ , for  $-z \leq t \leq -z+r$ , it can be at most twice as large, since  $g$  is increasing fastest at  $\lambda/2 - 2r$  and  $g(\lambda/2 - r) \leq 2g(\lambda/2 - 2r)$ . Hence the first integral on the right hand side of eq. (16) is at most

$$\int_{-z}^{-z+r} 2g(-z) \cdot q(z+t)^{q-1} dt \leq 2g(z) \cdot r^q, \quad (17)$$

while second integral on the right hand side of eq. (16) is at most

$$- \int_{-z+r}^{\lambda/2} G'(t) \cdot qr^{q-1} dt \leq qr^{q-1}, \quad (18)$$

since  $q(z+t)^{q-1}$  is a decreasing function in  $t$ . Putting eqs. (17) and (18) together with eq. (12), this gives us that  $-\frac{d}{dz}V(G)(z)$  is at most

$$V(G)(z) \cdot \left( G(\lambda/2)q(\lambda/2+z)^{q-1} + 2a \cdot g(z)r^q + qr^{q-1} \right) \leq 2g(z) \cdot (q + ar^q) + qr^{q-1},$$

which is also less than the bound from eq. (13).

Together, these two cases establish the claim.

Let  $F_0(z) = 0$  for all  $z$ , and  $F_{k+1} = V(F_k)$ . The operator  $V$  maps non-increasing functions to non-increasing functions, so all  $F_k$  are non-increasing. We know by [11, p.1078] that  $\lim_k F_{2k}$  and  $\lim_k F_{2k+1}$  exist, and equals the non-increasing functions  $F_A$  and  $F_B$  respectively. These functions are continuously differentiable, since  $V(F_A) = F_B$  and vice versa. If  $-F'_k \leq a \cdot g$ , then  $-F'_{k+1} \leq a \cdot g$ , while  $-F'_0 \leq a \cdot g$  holds trivially. By induction,  $-F'_k \leq a \cdot g$  for all  $k$ , and  $-F'_A \leq a \cdot g$ ,  $-F'_B \leq a \cdot g$  follows.

The next step is to show that  $\int_{-z}^{\lambda/2} \rho_A(t)dt$  is continuous in  $z$  on  $(-\lambda/2, \lambda/2)$ . It suffices to show that it is continuous on any closed subinterval  $I \subset (-\lambda/2, \lambda/2)$ . Since  $F'_A$  is bounded on  $I$ , there exists  $K_I$  such that for any  $x, y \in I$  we have

$$|F_A(x) - F_A(y)| < K_I \cdot |x - y| \quad (19)$$

We will let  $\varepsilon := \sqrt{|x - y|} \rightarrow 0$ . Suppose (without loss of generality) that  $x < y$  and  $y + \varepsilon \in I$ . We estimate the difference

$$\begin{aligned} & \left| \int_{-x}^{\lambda/2} q(x+t)^{q-1} F'_A(t) dt - \int_{-y}^{\lambda/2} q(y+t)^{q-1} F'_A(t) dt \right| \\ & \stackrel{(19)}{\leq} 2 \left| \int_{-x}^{-x+\varepsilon} K_I \cdot q(x+t)^{q-1} dt \right| + q \left| \int_{-y+\varepsilon}^{\lambda/2} \left( (x+t)^{q-1} - (y+t)^{q-1} \right) F'_A(t) dt \right| \\ & \leq O(\varepsilon^q) + 2|x-y| \cdot q(1-q) \cdot \left| \int_{-y+\varepsilon}^{\lambda/2} (y+t)^{q-2} F'_A(t) dt \right| = O(\varepsilon^q) + O(\varepsilon^q). \end{aligned}$$

Hence  $\int_{-z}^{\lambda/2} \rho_A^z(t)dt$  is continuous in  $z$ , and so is  $F'_B$ .

To establish the bound for  $\int_{-z}^{\lambda/2} \rho_A^z(t)dt$ , we use that  $-F'_A \leq a \cdot g$ . Then  $F_A$  satisfies the conditions necessary for eq. (14) to hold for  $z$  near  $-\lambda/2$  with  $G = F_A$ . For other  $z$ , note that the integrand is at most  $-F'_B(z)$ , for which the weaker bound  $a \cdot g$  suffices. In other words, for some constant  $b$  and any  $-\lambda/2 < z < \lambda/2$ , we have that

$$\int_{-z}^{\lambda/2} \rho_A^z(t)dt \leq b \max((\lambda/2 - z)_+^{q-1}, (z + \lambda/2)^{2q-1}),$$

while  $\int_{-\lambda/2}^{-z} \rho_A^z(t) dt = 0$ . Finally, for  $z > \lambda/2$ , the bound  $(z+t)^{q-1} \leq (z-\lambda/2)^{q-1}$  gives that  $\int \rho_A^z(t) dt \leq q(z-\lambda/2)^{q-1}$ . Setting  $\alpha = \max(a, b, q)$  gives the desired result.  $\blacksquare$

PROOF OF LEMMA 8. Assume that  $\phi \rightarrow u$  is an  $f_A$ -optimal move by Alice, and  $u \rightarrow v$  is an  $f_A$ - and  $f_B$ -optimal move by Bob. We will begin by showing that  $\mathbb{E}[|\Delta_t^k(v)| | f_A(u) = z]$  depends linearly (in  $z$ ) on  $\mathbb{E}[|\Delta_t^k(v)| | f_A(v) = z]$  for any fixed  $t$ .

Suppose we had some integer valued graph function<sup>2</sup>  $g$ , that  $u \rightarrow v$  was optimal with respect to both  $f_A$  and  $f_B$ , and that we wanted to find  $H(z) := \mathbb{E}[g(v) | f_A(u) = z]$  in terms of  $G(z) := \mathbb{E}[g(v) | f_B(v) = z]$ . How would we go about doing that?

If we let  $P^z(t) := \mathbb{P}(f_A(v) \leq t | f_A(u) = z)$  and  $\ell = f_A(u) + f_A(v)$  be the cost of the edge  $(u, v)$ , we can write  $H(z)$  the following way:

$$\begin{aligned} \mathbb{E}[g(v) | f_A(u) = z] &= \mathbb{E}[\mathbb{E}[g(v) | f_A(v)] | f_A(u) = z] \\ &= \int_{-\lambda/2}^{\lambda/2} \mathbb{E}[g(v) | f_A(v) = t, \ell = z + t] dP^z(t) = \int_{-\lambda/2}^{\lambda/2} G(t) dP^z(t), \end{aligned} \quad (20)$$

since the subtree rooted in  $v$  is independent of  $\ell$ . Thus  $H(z)$  is given by some functional (dependent on  $z$ ) applied to  $G$ , and hence  $H$  depends linearly on  $G$ .

Next we will show that  $L_A$  and  $L_B$  are indeed the right linear operators. Conditioning on  $f_A(u) = z$  is the same as conditioning on the optimal move  $v$  lying on the diagonal line  $\ell - f = z$  in the  $\ell f$ -square, and its marginal measure is precisely the denominator in eq. (7). Hence, by the same calculations as in eq. (20) and for  $g(v) = |\Delta_t^k(v)|$ ,

$$\frac{q(z + \lambda/2)_+^{q-1} F_A(\lambda/2) R^k(\lambda/2) + \int R^k(t) \rho_A^z(t) dt}{q(z + \lambda/2)_+^{q-1} F_A(\lambda/2) + \int \rho_A^z(t) dt} = \mathbb{E}[|\Delta_t^k(v)| | f_A(u) = z]$$

But  $R^k(\lambda/2) = 0$ , since Alice's optimal move from a vertex with game value  $\lambda/2$  will be to quit immediately. By eq. (7), this gives  $L_A(R_t^k)(z) = \mathbb{E}[|\Delta_t^k(v)| | f_A(u) = z]$ . Applying the same method one more time gives the desired result for the first part of the lemma. For the second part, we verify that (i)-(iii) hold.

- (i) The non-negative term  $\int \rho_A^z(t) dt$  is continuous in  $z$  by Lemma 7, and so is the positive term  $q(z + \lambda/2)^{q-1} F_A(\lambda/2)$ . Hence both numerator and denominator of eq. (8) are continuous, and the denominator is non-zero, so  $I_A$  is continuous.
- (ii) Both  $q(z + \lambda/2)^{q-1} F_A(\lambda/2)$  and  $\int \rho_A^z(t) dt$  are positive and finite when  $z \in (-\lambda/2, \lambda/2)$ , so  $I_A(z) < 1$  for such  $z$ .
- (iii) Using eq. (4), we see that for  $z$  near  $-\lambda/2$ ,

$$I_A(z) = \frac{O((z + \lambda/2)^{2q-1})}{(z + \lambda/2)^{q-1} + O((z + \lambda/2)^{2q-1})} = O((z + \lambda)^q),$$

so that  $\lim_{z \rightarrow -\lambda/2} I_A(z) = 0$ . Near  $\lambda/2$ ,

$$I_A(z) = \frac{\int \rho_A^z(t) dt}{q\lambda^{q-1} F_A(\lambda/2) + o(1) + \int \rho_A^z(t) dt} = 1 - \frac{1}{1 + \frac{\int \rho_A^z(t) dt}{q\lambda^{q-1} F_A(\lambda/2) + o(1)}},$$

so  $\lim_{z \rightarrow \lambda/2} I_A(z)$  will exist if  $\lim_{z \rightarrow \lambda/2} \int \rho_A^z(t) dt$  exists (even if it is infinite).

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<sup>2</sup>Here we write  $g(u)$  to denote the output of  $g$  when given the entire subtree rooted at  $u$  as its input.

Note that  $\int \rho_A^z(t)dt = \int \rho_A^z(t - z + \lambda/2)dt$ , as the support of  $\rho_A^z$  is  $[-z, \lambda/2] \subseteq [-\lambda/2, \lambda/2]$ , and translating by  $-z + \lambda/2$  gives a function with support  $[-\lambda/2, z] \subseteq [-\lambda/2, \lambda/2]$ . By eq. (3), we have that

$$\begin{aligned} \rho_A^z(t - z + \lambda/2) &= q(t + \lambda/2)^{q-1} \cdot F'_A(t - z + \lambda/2) \\ &\leq \begin{cases} \alpha q(t + \lambda/2)^{2q-2}, & t \leq 0 \\ K, & t > 0, \end{cases} \end{aligned}$$

for some constant  $K$  and all  $z$  sufficiently close to  $\lambda/2$ . Thus we have an upper bound on  $\rho_A^z(t - z + \lambda/2)$  which is independent of  $z$ . For  $q > 1/2$ , this upper bound is integrable and by dominated convergence it follows that

$$\lim_{z \rightarrow \lambda/2} \int \rho_A^z(t - z + \lambda/2)dt = \int \lim_{z \rightarrow \lambda/2} \rho_A^z(t - z + \lambda/2)dt = \int \rho_A^{\lambda/2}(t)dt < \infty.$$

Hence  $\lim_{z \rightarrow \lambda/2} \int \rho_A^z(t)dt$  exists (and is finite) for  $q > 1/2$ . For  $q \leq 1/2$ , we use eq. (2) of Lemma 7 to replace  $F'_A$ :

$$\int \rho_A^z(t)dt \geq \int_{-z}^{\lambda/2} q(t + \lambda/2)^{q-1} F_A(t) \cdot (F_B(\lambda/2) \cdot q(t + \lambda/2)^{q-1})dt.$$

This integral goes to  $\infty$  as  $z \rightarrow \lambda/2$ , since the pole  $(t + \lambda/2)^{2q-2}$  is not integrable. We conclude that  $\lim_{z \rightarrow \lambda/2} \int \rho_A^z(t)dt$  exists for all  $q$ , hence  $I_A(\lambda/2)$  is well defined.  $\blacksquare$

PROOF OF LEMMA 9.  $L_A$  is a substochastic operator<sup>3</sup>, and to be able to fully leverage this property we will factorize it into a stochastic operator that has almost all the structure of  $L_A$  and a substochastic operator that is a diagonal map. Start by defining the kernel  $\kappa_A^z(t)$ , as  $\rho_A^z$  normalized for  $(z, t) \in (-\lambda/2, \lambda/2)^2$ :

$$\kappa_A^z(t) := \frac{\rho_A^z(t)}{\int \rho_A^z(s)ds}. \quad (21)$$

Using this kernel, we write  $L_A(h)(z)$  as  $\int I_A(z)h(t)\kappa_A^z(t)dt$ . The factor  $I_A(z)$  does not depend on  $t$ , so it can be factored out of the integral. We therefore see  $L_A$  as the composition of the operators  $S_A$  and  $D_A$ , defined by

$$S_A(h)(z) := \int h(t)\kappa_A^z(t)dt \quad (22)$$

$$D_A(h)(z) := I_A(z) \cdot h(z). \quad (23)$$

For any function  $h$ ,  $\sup_t S_A(h)(t) \leq \sup_t h(t)$ , so  $\|S_A\| \leq 1$ . Similarly,  $\|S_B\| \leq 1$ .

As we want to show that  $\|L_B \circ L_A\| < 1$ , we factorize  $L_B \circ L_A$  into  $D_B \circ S_B \circ D_A \circ S_A$ . It then suffices to bound  $\|D_A\|$ ,  $\|D_B\|$  or  $\|D_B \circ S_B \circ D_A\|$  away from 1, since by the definition of the operator norm and using that  $\|S_A\|, \|S_B\| \leq 1$ , we have

$$\|L_B \circ L_A\| \leq \|D_B \circ S_B \circ D_A\| \leq \|D_A\| \cdot \|D_B\|.$$

The proof of the lemma will be divided into two cases, depending on whether  $I_A(\lambda/2) = I_B(\lambda/2) = 1$  or not.

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<sup>3</sup>A positive linear operator  $T$  given by  $T(h)(z) := \int h(t)\kappa^z(t)dt$  is said to be *stochastic* if  $\int \kappa^z(t)dt = 1$  for every  $z$  and *substochastic* if  $\int \kappa^z(t)dt \leq 1$  for every  $z$ .

CASE 1:  $I_A(\lambda/2) < 1$  or  $I_B(\lambda/2) < 1$

Assume without loss of generality that  $I_A(\lambda/2) < 1$ . Then  $I_A(z) < 1$  for all  $z$ . By Lemma 8,  $I_A$  is a continuous function on a closed interval, so it attains its maximum  $\theta$ , which must be less than 1. Thus we get that  $\|D_A\| \leq \theta < 1$ .

CASE 2:  $I_A(\lambda/2) = I_B(\lambda/2) = 1$

$D_B \circ S_B \circ D_A$  is an integral operator with kernel given by  $I_B(s)I_A(t)\kappa_B^s(t)$ . In order to show that this integral operator has norm less than 1, we will bound the integral of its kernel along the line  $s = z$ , where  $z \in (-\lambda/2, \lambda/2]$  is arbitrary but fixed. (We do not need to consider the case  $z = -\lambda/2$ , since  $I_B(-\lambda/2) < 1$ .)

We have a good upper bound on  $I_A$  and  $I_B$  on any closed set not containing  $z = \lambda/2$ . In particular, on  $[-\lambda/2, 0]$ . We therefore bound the total mass of  $\kappa_B^z$  on  $[0, \lambda/2]$ : Since  $I_B(\lambda/2) = 1$ , we know that  $\int_{-z}^{\lambda/2} \rho_B^z(t)dt \rightarrow \infty$  as  $z \rightarrow \lambda/2$ . But  $\int_0^{\lambda/2} \rho_B^z(t)dt \leq q\lambda^{q-1}$  for any  $z$ , so  $\int_0^{\lambda/2} \kappa_B^z(t)dt$  must vanish as  $z \rightarrow \lambda/2$ . Hence there exists  $0 < \eta < \lambda/2$  such that for all  $z > \eta$ ,

$$\int_{-z}^0 \kappa_B^z(t)dt > 1/2. \quad (24)$$

By (ii) of Lemma 8, we can find  $\delta > 0$ , such that when  $t \leq \eta$  we have

$$I_A(t) < 1 - \delta \quad \text{and} \quad I_B(t) < 1 - \delta. \quad (25)$$

Then, for  $-\lambda/2 \leq z \leq \eta$ , we apply the bound to  $I_B$  to get

$$\int I_B(z)I_A(t)\kappa_B^z(t)dt \leq I_B(z) \cdot \int \kappa_B^z(t)dt \stackrel{(25)}{<} 1 - \delta,$$

while for  $\eta < z \leq \lambda/2$  we apply it to  $I_A$

$$\int I_B(z)I_A(t)\kappa_B^z(t)dt \stackrel{(25)}{<} \int_{-z}^0 (1 - \delta)\kappa_B^z(t)dt + \int_0^{\lambda/2} \kappa_B^z(t)dt \stackrel{(24)}{<} 1 - \delta/2.$$

This means that the weight along each line of  $D_B \circ S_B \circ D_A$  is at most  $1 - \delta$  when  $z \leq \eta$ , and at most  $1 - \delta/2$  otherwise. Hence  $\|D_B \circ S_B \circ D_A\| \leq 1 - \delta/2$ .

As we have finished the proof in these two cases, we can conclude that  $\|L_B \circ L_A\| < 1$ . ■

Now that we have Lemmas 7 to 9, we can proceed with the proof of Proposition 6.

PROOF OF PROPOSITION 6. We will use  $\|L_B \circ L_A\| < 1$  to construct suitable  $\psi_t$ . In a slight abuse of notation, let 1 denote both the natural number 1 and the function on  $[-\lambda/2, \lambda/2]$  with constant value 1. Let the functions  $\psi_t$  (for any  $0 \leq t \leq 2\lambda$  and some large constants  $K, m > 0$  to be determined later) be defined by

$$\psi_t := K \exp(mt) \cdot \sum_{k=0}^{\infty} (L_B \circ L_A)^k(1). \quad (26)$$

By Lemma 9,  $\|L_B \circ L_A\| < 1$ , so the above series is absolutely convergent, whence its norm is at most  $K \exp(mt)/(1 - \|L_B \circ L_A\|)$ . Furthermore,  $K \exp(mt)$  is a constant with respect to  $z$ , so if we apply the operator  $L_B \circ L_A$  to  $\psi_t$  we can apply it termwise to the sum in eq. (26).

$$L_B \circ L_A(\psi_t)(z) = K \exp(mt) \cdot \sum_{k=1}^{\infty} (L_B \circ L_A)^k(1)(z) = \psi_t(z) - K \exp(mt). \quad (27)$$

Crucially,  $L_B \circ L_A(\psi_t)$  is less than  $\psi_t$ , and with a sizeable margin. We will do induction on even  $k$  to establish the main claim. Fix some  $-\lambda/2 \leq z \leq \lambda/2$  and  $0 \leq t \leq 2\lambda$ , and consider the first two moves of the game conditioned on  $f_A(\phi) = z$ .

Let  $u$  be Alice's optimal move from the root (if such a move exists, the result holds trivially otherwise), and  $v_0$  the vertex she expects Bob to move to after that. Assume there are  $n$  of Bob's move options from  $u$  that are mistakes costing at most  $t$ . Let  $v_i$ ,  $1 \leq i \leq n$ , be the vertices those moves lead to,  $t_i$  the cost of each mistake,  $\ell_i$  the cost of the edge  $(u, v_i)$ , and let  $f_i := f_A(v_i)$ . Note that  $t_i$ ,  $v_i$ ,  $f_i$  and  $n$  are random variables, with distribution conditional on  $f_A(\phi) = z$ .

For the base case  $k = 2$ , the root has at most one child  $u$  in  $\Delta_t^2(\phi)$ . Its expected number of children, conditioned on any value of  $f_A(u)$ , is at most  $1 + \lambda^q$ , so we let  $K = 2(2 + \lambda^q)$ . Then  $R_t^2 \leq 2 + \lambda^q < \psi_t$ , establishing the base case.

Next, assume  $R_s^k < \psi_s$  for some even  $k > 2$  and all  $0 \leq s \leq 2\lambda$ . We want to show that  $R_t^{k+2} < \psi_t$  as well, and to do that we will bound the expected size of  $\Delta_t^{k+2}$ . The tree  $\Delta_t^{k+2}$  can be written as an edge-disjoint union of copies of  $\Delta$  in the following way:

$$\Delta_t^{k+2}(\phi) = \Delta_t^2(\phi) \cup \Delta_t^k(v_0) \cup \bigcup_{i=1}^n \Delta_{t-t_i}^k(v_i). \quad (28)$$

We already have a bound for  $\Delta_t^2(\phi)$ , and we continue by bounding the conditional expected sizes of  $\Delta_t^k(v_0)$  and  $\bigcup_{i=1}^n \Delta_{t-t_i}^k(v_i)$ . For  $\Delta_t^k(v_0)$ , by the definition of  $R_t^k$  and Lemma 8,

$$\begin{aligned} \mathbb{E}[|\Delta_t^k(v_0)| | f_A(\phi) = z] &= L_B \circ L_A(R_t^k)(z) \\ &< L_B \circ L_A(\psi_t)(z), \text{ since } L_B, L_A \text{ positive operators} \\ &\stackrel{(27)}{=} \psi_t(z) - K \exp(mt). \end{aligned} \quad (29)$$

Finally, we bound the expected size of the union of the trees  $\Delta_{t-t_i}^k(v_i)$ . To do this we condition first on the random variables  $n$ ,  $f_i$  and  $t_i$  and then on the event  $f_A(\phi) = z$ , so that the first conditional expectation is itself a random variable.

$$\begin{aligned} \mathbb{E}\left[\left|\bigcup_{i=1}^n \Delta_{t-t_i}^k(v_i)\right| \middle| f_A(\phi) = z\right] &= \mathbb{E}\left[\mathbb{E}\left[\left|\bigcup_{i=1}^n \Delta_{t-t_i}^k(v_i)\right| \middle| n, t_i, f_i, 1 \leq i \leq n\right] \middle| f_A(\phi) = z\right] \\ &= \mathbb{E}\left[\sum_{i=1}^n R_{t-t_i}^k(f_i) \middle| f_A(\phi) = z\right], \end{aligned} \quad (30)$$

since the subtree rooted in  $v_i$ , conditioned on  $f_A(v_i)$ , is independent of  $f_A(\phi)$ . By the induction hypothesis with  $s = t - t_i$ ,

$$\sum_{i=1}^n R_{t-t_i}^k(f_i) \leq \sum_{i=1}^n \psi_{t-t_i}(f_i).$$

Let  $\sigma_A$  be the Poisson random measure generated by  $\mu_A$ . It is a sum of Dirac measures, each corresponding to a point in the  $\ell f$ -square. Among these points,  $(\ell_i, f_i)$ ,  $1 \leq i \leq n$ , are exactly those that lie in the diagonal strip  $z < \ell - f \leq z + t$ . Note that for any bounded  $\mu_A$ -measurable function  $h$ , we have that  $\mathbb{E}[\int h d\sigma_A] = \int h d\mu_A$  (which can be seen by approximating  $h$  by simple functions). The expression 30 is then at most

$$\mathbb{E}\left[\sum_{i=1}^n \psi_{t-t_i}(f_i) \middle| f_A(\phi) = z\right] = \mathbb{E}\left[\int_{z < \ell - f \leq z + t} \psi_{z+t-\ell+f}(f) d\sigma_A(\ell, f)\right]$$

$$= \int_{z < \ell - f \leq z+t} \psi_{z+t-\ell+f}(f) d\mu_A(\ell, f). \quad (31)$$

We have that  $\psi_s \leq K \exp(ms)/(1 - \|L_B \circ L_A\|)$  for all  $s$ , so if we use this bound and integrate first along diagonals  $\ell - f = x$  for  $x$  fixed, and then integrate over  $x$ , we get

$$\begin{aligned} (31) &\leq \frac{K}{1 - \|L_B \circ L_A\|} \cdot \int_z^{z+t} J_A^x \exp(m(z+t-x)) dx \\ &\stackrel{(5)}{\leq} \frac{K \exp(mt)}{1 - \|L_B \circ L_A\|} \cdot \alpha \int_z^\infty [(x + \lambda/2)^{q-1} + |x - \lambda/2|^{q-1}] \exp(m(z-x)) dx \\ &\leq K \exp(mt) \cdot \varepsilon_m, \end{aligned} \quad (32)$$

where  $\varepsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ , and does not depend on  $k, t$  or  $z$ . We now have a bound on the expected size of each term in the right hand side of eq. (28). The bounds from eqs. (29) and (32) give that

$$\begin{aligned} R_t^{k+2}(z) &= \mathbb{E}[|\Delta_t^2(\phi)| | f_A(\phi) = z] + \mathbb{E}[|\Delta_t^k(v)| | f_A(\phi) = z] + \sum_i \mathbb{E}[|\Delta_{t-t_i}^k(v_i)| | f_A(\phi) = z] \\ &< 2 + \lambda^q + \psi_t(z) - K \exp(mt) + K \exp(mt) \varepsilon_m. \end{aligned} \quad (33)$$

We pick  $m$  large enough so that  $\varepsilon_m < 1/2$ . The expression (33) is then less than  $\psi_t(z)$ , and the inductive step is complete. Hence  $R_t^k < \psi_t$  for all even  $k$  and  $0 \leq t \leq 2\lambda$ .  $\blacksquare$

**PROOF OF PROPOSITION 3.** By Lemma 4, the game path  $P$  is  $(\phi, 2\lambda)$ -reasonable, and is therefore contained in the tree  $\Delta_{2\lambda}(\phi)$  of all  $(\phi, 2\lambda)$ -reasonable paths. By Proposition 6,  $\Delta_{2\lambda}(\phi)$  is almost surely finite, and hence the game finishes after finitely many steps. Thus  $P$  is the finite path  $\phi = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_N$  for some  $u_i$ 's. For  $1 \leq i \leq N$ , let  $\ell_i = \ell(v_{i-1}, v_i)$ . Let  $L$  be the total payoff for Alice. (The total payoff for Bob is then  $-L$ .)

Recall that  $f_A(u_{i-1}) \leq \ell_i - f_A(u_i)$ , with equality if  $u_{i-1} \rightarrow u_i$  is  $f_A$ -optimal (which is always the case if  $i$  is odd). Using these inequalities along  $P$  gives us

$$f_A(\phi) = \ell_1 - f_A(u_1) \geq \ell_1 - \ell_2 + f_A(u_2) = \dots \geq \sum_{i=1}^N (-1)^{i+1} \ell_i + (-1)^N f_A(u_N)$$

$$\text{CLAIM: } \sum_{i=1}^N (-1)^{i+1} \ell_i + (-1)^N f_A(u_N) \geq -L$$

We divide the proof of the claim into cases, depending on if  $N$  is even or odd. If  $N$  is even, then Alice is the one that quits, and  $f_A(u_N) = \lambda/2$ . Hence  $L = -\lambda/2 + \sum_{i=1}^N (-1)^i \ell_i = -f_A(u_N) + \sum_{i=1}^N (-1)^i \ell_i$ . If on the other hand  $N$  is odd, then Bob is the one that quits, and (like for every vertex)  $f_A(u_N) \leq \lambda/2$ . Hence  $L = \lambda/2 + \sum_{i=1}^N (-1)^i \ell_i \geq f_A(u_N) + \sum_{i=1}^N (-1)^i \ell_i$ .

So in either case the claim is true, and thus  $f_A(\phi) \geq -L$ . By a similar argument (with the roles of Alice and Bob reversed, and  $\delta$  instead measuring how far Alice deviates from what is  $f_B$ -optimal) we have that  $-f_B(\phi) \geq L$ .

Thus  $f_A(\phi) \geq -L \geq f_B(\phi)$ . By the choice of  $f_A$  and  $f_B$ , we know that  $f_A(\phi) \leq f_B(\phi)$ , so we have that  $f_A(\phi) = f_B(\phi)$ . For any other  $u \in V(T_\lambda^q)$ , the subtree rooted in  $u$  has the same distribution as the whole  $T_\lambda^q$ , so a similar argument gives that  $f_A(u) = f_B(u)$ . Hence  $f_A = f_B$ . Since  $f_A$  and  $f_B$  are the maximum and minimum, respectively, in the lattice ordering of all valuations, this implies that the valuation is unique.  $\blacksquare$

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